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Compatible Smooth Interpolation in Triangles

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Boolean sum smooth interpolation to boundary data on a triangle is described. Sufficient conditions are given so that the functions when pieced together form a $C^{N-1}(\Omega)$ function over a triangular subdivision of a polygonal region Ω and the precision sets of the interpolation functions are derived. The interpolants are modified so that the compatibility conditions on the function which is interpolated can be removed and a C^1 interpolant is used to illustrate the theory. The generation of interpolation schemes for discrete boundary data is also discussed.

1. INTRODUCTION

This note presents a method of removing the compatibility conditions from the smooth rational Boolean sum interpolants described by Barnhill, Birkhoff, and Gordon [2]. The resulting interpolants match a function $F \in C^{N-1}(\partial T)$ and its first $N-1$ normal derivatives along the boundary ∂T of a triangle T. The interpolants can be used to define a piecewise function which is $C^{N-1}(\Omega)$ over a triangulated polygon Ω.

For algebraic simplicity, the triangle T with vertices at $V_1 = (0, 1)$, $V_2 = (1, 0)$, and $V_3 = (0, 0)$ is considered, where the side opposite the vertex V_k is denoted by E_k. Any other triangle can be obtained by an affine

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transformation of this "standard" triangle. Hermite interpolation projectors P_k, $k = 1, 2, 3$, are defined so that on two sides of the triangle T, the function $P_kF(x, y)$ interpolates $F \in C^{N-1}(\partial T)$ and its first $N - 1$ directional derivatives along the line through (x, y) parallel to the third side E_k. Such a function will interpolate all partial derivatives up to order $N - 1$ on the two sides, since tangential derivatives along the sides are automatically given. Boolean sum operators of these projectors are defined by

$$P_i \oplus P_j = P_i + P_j - P_iP_j, \qquad i \neq j, \quad 1 \leqslant i, \quad j \leqslant 3. \tag{1.1}$$

Barnhill and Mansfield [4] show that the function $(P_i \oplus P_j)F$ interpolates $F \in C^{N-1}(\partial T)$, and all its partial derivatives of order $N - 1$ and less on ∂T, provided that certain derivatives of F are compatible at a vertex. In Section 2, we show how these compatibility conditions may be removed, whilst preserving the interpolation properties and precision set of the resulting interpolant. This is an alternative approach to that proposed by Mansfield [7]. The precision set is the set of polynomials for which the interpolant is exact, and it is important in that it indicates the order of accuracy of the interpolant.

Smooth interpolation functions have applications to computer aided geometric design and finite element analysis. Finite element analysis involves the piecewise approximation of the solution of a variational problem. One source of such a variational problem is an elliptic boundary value problem. The domain is divided into elements, e.g., triangles, and the finite element basis functions can then be generated by taking the boundary data over the subtriangles as polynomials which interpolate certain linear functionals along a side. This possibility is discussed in Section 4. An additional application of smooth interpolants to finite element analysis is that data along the boundary of a polygonal region can be matched exactly by smooth interpolants.

The mathematics of computer aided geometric design involves the interpolation and approximation of curves and surfaces. Smooth interpolation functions can be used to blend together given space curves. These interpolants become finite dimensional if the curves depend on finitely many parameters. Desirable properties for computer aided geometric design are: a local basis, smoothness (i.e., C^0 or C^1), and computational ease. In addition, the defining parameters must have geometric significance for the designer. For example, the removal of certain "twist" parameters that are sometimes of uncertain geometric significance is discussed in [6]. An exposition of smooth interpolants and of finite dimensional interpolants for computer aided geometric design is given in [1].

2. Hermite Boolean Sum Interpolation to Boundary Data

The Hermite interpolation projectors on the standard triangle T are defined by

$$P_1F = \sum_{i=0}^{N-1} \phi_i\Big(\frac{x}{1-y}\Big)(1-y)^i F_{i,0}(0,y) + \sum_{i=0}^{N-1} \psi_i\Big(\frac{x}{1-y}\Big)(1-y)^i F_{i,0}(1-y,y), \tag{2.1}$$

$$P_2F = \sum_{i=0}^{N-1} \phi_i\Big(\frac{y}{1-x}\Big)(1-x)^i F_{0,i}(x,0) + \sum_{i=0}^{N-1} \psi_i\Big(\frac{y}{1-x}\Big)(1-x)^i F_{0,i}(x,1-x), \tag{2.2}$$

$$P_3F = \sum_{i=0}^{N-1} \phi_i\Big(\frac{x}{x+y}\Big)(x+y)^i \left(\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right]^i F\right)(0,x+y) + \sum_{i=0}^{N-1} \psi_i\Big(\frac{x}{x+y}\Big)(x+y)^i \left(\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right]^i F\right)(x+y,0), \tag{2.3}$$

where the $\phi_i(t)$ and $\psi_i(t) = (-1)^i\,\phi_i(1-t)$ are the cardinal basis functions for Hermite two-point Taylor interpolation on the interval [0, 1]. Thus

$$\phi_i^{(j)}(0) = \delta_{ij} \qquad \text{and} \qquad \phi_i^{(j)}(1) = 0, \qquad 0 \leqslant i,j \leqslant N-1. \tag{2.4}$$

It is convenient to express these projectors as

$$P_i = P_i^j + P_i^k, \qquad i \neq j \neq k \neq i, \tag{2.5}$$

where P_i^l denotes the part of the projector involving the side E_l. Since $P_i^kP_j = P_i^k$, the Boolean sum of two projectors can then be expressed as

$$P_i \oplus P_j = P_i + P_j - P_iP_j = P_i^j + P_j - P_i^jP_j\,. \tag{2.6}$$

Boolean sum interpolation properties. The following theorem on Boolean sum interpolation is Lemma 4.2 of [4], but with a modified proof. At the end of Section 2, we show how the Boolean sum function can be modified so that the compatibility conditions (2.7) can be relaxed.

Theorem 2.1. *The Boolean sum functions*

$$(P_i \oplus P_j)F = (P_i + P_j - P_iP_j)F, \qquad i \neq j; \quad i,j = 1,2,3,$$

interpolate $F \in C^{N-1}(\partial T)$ and its derivatives of order $N-1$ and less on ∂T, provided that F satisfies the compatibility conditions

$$\left(\frac{\partial^{m+n}F}{\partial s_i^m \partial s_j^n}\right)(V_k) = \left(\frac{\partial^{n+m}F}{\partial s_j^n \partial s_i^m}\right)(V_k), \qquad m, n < N;\ m + n \geqslant N, \tag{2.7}$$

where V_k is the vertex with adjacent sides E_i and E_j, and $\partial/\partial s_l$ denotes differentiation along the side E_l.

Proof. It is sufficient to consider the case $(P_1 \oplus P_2)F$ as the other cases then follow by affine transformation and symmetry. First, the interpolation properties hold on $x = 0$ and $1 - x - y = 0$ since

$$F - (P_1 \oplus P_2)F = (I - P_1)(I - P_2)F,$$

where I is the identity operator, and $I - P_1$ and its partial derivatives of order $N - 1$ and less are null on those sides. Second, from (2.6),

$$F - (P_1 \oplus P_2)F = (I - P_2)F - P_1^2(I - P_2)F$$

and hence we require that

$$\left(\frac{\partial^j P_1^2(I - P_2)F}{\partial y^j}\right)(x, 0) = 0, \qquad 0 \leqslant j \leqslant N - 1,$$

for the interpolation properties to hold on $y = 0$. Now

$$P_1^2(I - P_2)F = \sum_{i=0}^{N-1} \phi_i\left(\frac{x}{1-y}\right)(1 - y)^i \left[F_{i,0}(0, y) - \left(\frac{\partial^i P_2 F}{\partial x^i}\right)(0, y)\right]. \tag{2.8}$$

Also,

$$\left(\frac{\partial^{k+i} P_2 F}{\partial y^k \partial x^i}\right)(0, 0) = \left(\frac{\partial^{i+k} P_2 F}{\partial x^i \partial y^k}\right)(0, 0)$$
$$= \left(\frac{\partial^i F_{0,k}}{\partial x^i}\right)(0, 0), \qquad 0 \leqslant i, k \leqslant N - 1,$$

where the change of order of differentiation is permissible as, for $F \in C^{N-1}(\partial T)$, it can be shown that P_2F is $N - 1$ times continuously differentiable at (0, 0). Thus, differentiation of (2.8) by Leibniz' rule gives

$$\left(\frac{\partial^j P_1^2(I - P_2)F}{\partial y^j}\right)(x, 0)$$
$$= \sum_{k=1}^{j} \sum_{i=N-k}^{N-1} \binom{j}{k} \left(\frac{\partial^{j-k}}{\partial y^{j-k}} \phi_i\left(\frac{x}{1-y}\right)(1 - y)^i\right)(x, 0)$$
$$\times \left[\left(\frac{\partial^k}{\partial y^k}\left(\frac{\partial^i F}{\partial x^i}\right)\right)(0, 0) - \left(\frac{\partial^i}{\partial x^i}\left(\frac{\partial^k F}{\partial y^k}\right)\right)(0, 0)\right], \tag{2.9}$$

where the original summation $\sum_{i=0}^{N-1} \sum_{k=0}^{j}$ has been reduced to that in (2.9), since the cross derivatives are compatible for $0 \leqslant i+k \leqslant N-1$ if $F \in C^{N-1}(\partial T)$. The requirement that (2.9) be zero for all $1 \leqslant j \leqslant N-1$ is true if and only if

$$\left(\frac{\partial^k}{\partial y^k}\left(\frac{\partial^i F}{\partial x^i}\right)\right)(0,0) = \left(\frac{\partial^i}{\partial x^i}\left(\frac{\partial^k F}{\partial y^k}\right)\right)(0,0), \quad i,k < N;\, i+k \geqslant N, \tag{2.10}$$

which completes the proof of the theorem.

Continuity at vertices. The previous theorem indicates the behavior of the Boolean sum functions along ∂T. The following theorem gives sufficient conditions that the apparent singularities at two of the vertices of T are removable and also that the Boolean sum function be $N-1$ times continuously differentiable.

THEOREM 2.2. *Let the derivative values of F in $(P_i \oplus P_j)F$, as functions of their single variables, be $N-1$ times continuously differentiable on the sides of T and the function values of F be N times continuously differentiable on the sides of T. Then $(P_i \oplus P_j)F$ is $N-1$ times continuously differentiable on T excluding the vertices V_i and V_j, where, for $F \in C^{N-1}(\partial T)$, it is $N-1$ times continuously differentiable as the limit to the vertices is approached from T.*

Proof. It is sufficient to consider the case

$$\begin{aligned}(P_1 \oplus P_2)F &= P_2F + P_1{}^2(I-P_2)F \\ &= \sum_{i=0}^{N-1} \phi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x,0) \\ &\quad + \sum_{i=0}^{N-1} \psi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x,1-x) \\ &\quad + \sum_{i=0}^{N-1} \phi_i\left(\frac{x}{1-y}\right)(1-y)^i\left[F_{i,0}(0,y) - \left(\frac{\partial^i P_2 F}{\partial x^i}\right)(0,y)\right]. \end{aligned} \tag{2.11}$$

Since the boundary functions are $N-1$ times continuously differentiable with respect to their single variables x or y on $[0,1]$, then (2.11) is also $N-1$ times continuously differentiable on T excluding the vertices $V_1 = (0,1)$ and $V_2 = (1,0)$, where, for $(x,y) \in T$, the functions ϕ and ψ in (2.11) have singularities.

Consider first $V_2 = (1, 0)$, where from (2.11), it follows that

$$\lim_{(x,y)=(1,0)} \frac{\partial^{m+n}(P_1 \oplus P_2)F}{\partial x^m \partial y^n} = \lim_{(x,y)=(1,0)} \frac{\partial^{m+n} P_2 F}{\partial x^m \partial y^n}, \quad 0 \leqslant m + n \leqslant N - 1. \tag{2.12}$$

The differentiability of the function and derivative values of F in P_2F implies that, for $F \in C^{N-1}(\partial T)$, they have the following Taylor expansions about $x = 1$.

$$F_{0,i}(x, 0) = \sum_{j=0}^{N-1-i} (x-1)^{(j)} F_{j,i}(1, 0) + \int_1^x (x - \tilde{x})^{(N-1-i)} \frac{d^{N-i} F_{0,i}(\tilde{x}, 0)}{d\tilde{x}^{N-i}} \, d\tilde{x}, \tag{2.13}$$

$$F_{0,i}(x, 1-x) = \sum_{j=0}^{N-1-i} (x-1)^{(j)} \left(\frac{d^j F_{0,i}(x, 1-x)}{dx^j} \right)_{x=1} + \int_1^x (x - \tilde{x})^{(N-1-i)} \frac{d^{N-i} F_{0,i}(\tilde{x}, 1-\tilde{x})}{d\tilde{x}^{N-i}} \, d\tilde{x}, \tag{2.14}$$

where

$$(x-1)^{(j)} = \frac{(x-1)^j}{j!}, \qquad \text{etc.}$$

Now P_2 is exact for the function

$$\sum_{i+j<N} (x-1)^{(i)} y^{(j)} F_{i,j}(1, 0),$$

[see proof of Theorem 2.3, Eq. (2.21)], and the integral remainder terms in (2.13) and (2.14) are zero for such a function. Thus substitution of the Taylor expansions into Eq. (2.2) for P_2F gives

$$\begin{aligned} P_2F = {} & \sum_{i+j<N} (x-1)^{(i)} y^{(j)} F_{i,j}(1, 0) \\ & + \sum_{i=0}^{N-1} \phi_i \left(\frac{y}{1-x} \right) (1-x)^i \int_1^x (x - \tilde{x})^{(N-1-i)} \frac{d^{N-i} F_{0,i}(\tilde{x}, 0)}{d\tilde{x}^{N-i}} \, d\tilde{x} \\ & + \sum_{i=0}^{N-1} \psi_i \left(\frac{y}{1-x} \right) (1-x)^i \int_1^x (x - \tilde{x})^{(N-1-i)} \frac{d^{N-i} F_{0,i}(\tilde{x}, 1-\tilde{x})}{d\tilde{x}^{N-i}} \, d\tilde{x}. \end{aligned} \tag{2.15}$$

We note that the behavior of the singular terms is like

$$\phi_i \left(\frac{y}{1-x} \right) (1-x)^N \qquad \text{and} \qquad \psi_i \left(\frac{y}{1-x} \right) (1-x)^N.$$

Thus, since $0 \leqslant y/(1-x) \leqslant 1$ for $(x, y) \in T$, it is simple to show by differentiating (2.15), and using the continuity and hence boundedness of the derivatives, that

$$\lim_{(x,y)\to(1,0)} \frac{\partial^{m+n} P_2 F}{\partial x^m \partial y^n} = F_{m,n}(1, 0), \quad 0 \leqslant m+n \leqslant N-1; (x, y) \in T, \tag{2.16}$$

which by (2.12) is desired result at the vertex $V_2 = (1, 0)$. At $V_1 = (0, 1)$, since

$$(P_1 \oplus P_2)F = P_1 F + (I - P_1) P_2 F,$$

the dual argument applied to $P_1 F$ and $P_1[P_2 F]$ gives that

$$\lim_{(x,y)\to(0,1)} \frac{\partial^{m+n}(P_1 \oplus P_2)F}{\partial x^m \partial y^n} = F_{m,n}(0, 1), \qquad 0 \leqslant m+n \leqslant N-1; (x, y) \in T. \tag{2.17}$$

Remark. Weaker conditions on $F(0, x)$ and $F(x, 1-x)$ are that their $(N-1)$st derivative be absolutely continuous and their Nth derivative be bounded in the L_p norm, $p > 1$, on the interval $[1-\epsilon, 1]$.

Theorems 2.1 and 2.2 give sufficient conditions that the interpolation functions can be pieced together to form a $C^{N-1}(\Omega)$ function over a triangulated polygon Ω. Theorem 2.2 holds for boundary functions which are polynomials.

Precision. The following simple lemma is useful in establishing the set of polynomials for which the Boolean sum interpolants are exact.

LEMMA 2.1. *Let f and g be functions of one and two variables, respectively. Then*

$$P_i[f(\xi_i)\, g(x, y)] = f(\xi_i)\, P_i[g(x, y)], \qquad i = 1, 2, 3, \tag{2.18}$$

where $\xi_i = 0$ is the side E_i of ∂T parallel to the projector P_i, i.e.,

$$\xi_1 \equiv y, \qquad \xi_2 \equiv x, \qquad \textit{and} \qquad \xi_3 \equiv 1 - x - y. \tag{2.19}$$

Proof. Substitution of $F(x, y) = f(\xi_i)\, g(x, y)$ into the function $P_i F$ gives the desired result.

THEOREM 2.3. *The operator $P_i \oplus P_j$ is exact for the set of monomials*

$$\xi_j^m \xi_i^n, \qquad \begin{cases} m \geqslant 0, \text{ for } 0 \leqslant n \leqslant 2N-1, \\ 2N \leqslant m+n \leqslant 3N-1, \text{ for } 2N \leqslant n \leqslant 3N-1. \end{cases} \tag{2.20}$$

Proof. By affine transformation and symmetry it is sufficient to consider the case $P_1 \oplus P_2$.

First, the operator $P_1 \oplus P_2$ has at least the precision set of P_2 since

$$P_1 \oplus P_2 = P_2 + P_1(I - P_2)$$

and $I - P_2$ is the null operator on that precision set. Now from Lemma 2.1, $P_2 x^m y^n = x^m P_2 y^n$ for all m, and by the precision of the Hermite projector P_2, it follows that $P_2 y^n = y^n$ for $0 \leqslant n \leqslant 2N - 1$. Thus

$$P_2 x^m y^n = x^m y^n, \qquad m \geqslant 0; \quad 0 \leqslant n \leqslant 2N - 1. \tag{2.21}$$

Second, using Lemma 2.1, we have that

$$(P_1 \oplus P_2)\, x^m y^n = y^n P_1 x^m + x^m P_2 y^n - P_1[x^m P_2 y^n]. \tag{2.22}$$

Now for $n \geqslant N$,

$$P_2 y^n = \sum_{i=0}^{N-1} \psi_i \left(\frac{y}{1-x}\right) (1-x)^i \frac{n!}{(n-i)!} (1-x)^{n-i}$$

and each $\psi_i(t)$ is a polynomial of degree $2N - 1$ with a factor t^N. Thus, by cancellation for $n \geqslant 2N$, $x^m P_2 y^n$ is a polynomial in x and y which is of degree at most $m + n - N$ in the variable x. Hence, using the precision of P_1, it follows that $P_1[x^m P_2 y^n] = x^m P_2 y^n$ for $m + n - N \leqslant 2N - 1$ and $n \geqslant 2N$. Substituting in (2.22), and since $P_1 x^m = x^m$ for $0 \leqslant m \leqslant 2N - 1$, we have that

$$(P_1 \oplus P_2)\, x^m y^n = x^m y^n, \qquad m + n \leqslant 3N - 1, \quad 2N \leqslant n \leqslant 3N - 1. \tag{2.23}$$

It follows from Theorem 2.3 that Boolean sum interpolation on a general triangle is exact for all polynomials of degree $3N - 1$ or less, since this set is affine invariant. More precisely we have

COROLLARY 2.1. *Let P_i and P_j be two Hermite projectors along parallels to the sides E_i and E_j of a general triangle in the (x, y) plane. Let $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ be the affine transformation which transforms this triangle onto the standard triangle in the (ξ, η) plane, with the side E_i on the ξ axis and E_j on the η axis. Then the Boolean sum interpolant $(P_i \oplus P_j)F$ is exact for the set of polynomials*

$$\eta^m \xi^n, \qquad \begin{cases} m \geqslant 0, \text{ for } 0 \leqslant n \leqslant 2N - 1, \\ 2N \leqslant m + n \leqslant 3N - 1, \text{ for } 2N \leqslant n \leqslant 3N - 1. \end{cases} \tag{2.24}$$

Removal of compatibility conditions. Without loss of generality the Boolean sum function $(P_1 \oplus P_2)F$ is considered. By Theorem 2.1, the remainder function

$$R = F - (P_1 \oplus P_2)F$$

is zero, and has zero derivatives of order $N-1$ and less, on the sides of the triangle T, except on $y = 0$, where

$$R_{0,j}(x, 0) = -\sum_{k=1}^{j} \sum_{i=N-k}^{N-1} \binom{j}{k} \left(\frac{\partial^{j-k}}{\partial y^{j-k}} \phi_i \left(\frac{x}{1-y}\right) (1-y)^i\right)(x, 0)$$
$$\times \left[\left(\frac{\partial^{k+i}F}{\partial y^k \partial x^i}\right)(0, 0) - \left(\frac{\partial^{i+k}F}{\partial x^i \partial y^k}\right)(0, 0)\right], \qquad 1 \leqslant j \leqslant N-1.$$

It is easily shown that $R \in C^{N-1}(\partial T)$ and that at the vertex $V_1 = (0, 1)$, R satisfies the compatibility requirements (2.7) of Theorem 2.1. Thus the Boolean sum functions

$$(P_2 \oplus P_3)R \qquad \text{and} \qquad (P_3 \oplus P_2)R$$

interpolate the function R and its derivatives of order $N-1$ and less on the sides of T. Either of these functions when added to $(P_1 \oplus P_2)F$ removes the compatibility conditions of Theorem 2.1. By affine transformation and symmetry, we have thus proved the following theorem.

THEOREM 2.4. *The functions*

$$(P_i \oplus P_j)F + (P_j \oplus P_k)R, \qquad i \neq j \neq k \neq i, \tag{2.25}$$

$$(P_i \oplus P_j)F + (P_k \oplus P_j)R, \tag{2.26}$$

where

$$R = F - (P_i \oplus P_j)F, \tag{2.27}$$

interpolate $F \in C^{N-1}(\partial T)$ *and its derivatives of order* $N-1$ *and less on* ∂T.

We note that the modified interpolant reduces to $(P_i \oplus P_j)F$ when the compatibility conditions of Theorem 2.1 are satisfied.

The precision set of Theorem 2.3 is true for the modified interpolants (2.25) and (2.26), since the corrective functions are zero for all polynomial F. Theorem 2.2 is also valid for the modified interpolants.

3. C^1 INTERPOLATION TO BOUNDARY DATA

In this section the theory of Section 2 is illustrated with the case $N = 2$ of Boolean sum functions which interpolate $F \in C^1(\partial T)$ and its first derivatives on ∂T. The basis functions for Hermite two-point Taylor interpolation on [0, 1] are then defined by

$$\begin{aligned} \phi_0(t) &= (t-1)^2(2t+1), & \phi_1(t) &= (t-1)^2 t, \\ \psi_0(t) &= t^2(-2t+3), & \psi_1(t) &= t^2(t-1). \end{aligned} \tag{3.1}$$

The Boolean sum function of P_1 and P_2 is

$$\begin{aligned} (P_1 \oplus P_2)F = {} & \sum_{i=0}^{1} \phi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x, 0) \\ & + \sum_{i=0}^{1} \psi_i\left(\frac{y}{1-x}\right)(1-x)^i F_{0,i}(x, 1-x) \\ & + \sum_{i=0}^{1} \phi_i\left(\frac{x}{1-y}\right)(1-y)^i \left[F_{i,0}(0, y) - \left(\frac{\partial^i P_2 F}{\partial x^i}\right)(0, y)\right], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} (P_2 F)(0, y) = {} & \phi_0(y)\, F(0, 0) + \phi_1(y)\, F_{0,1}(0, 0) \\ & + \psi_0(y)\, F(0, 1) + \psi_1(y)\, F_{0,1}(0, 1), \\ \left(\frac{\partial P_2 F}{\partial x}\right)(0, y) = {} & y[\phi_0'(y)\, F(0, 0) + \phi_1'(y)\, F_{0,1}(0, 0) + \psi_0'(y)\, F(0, 1) \\ & + \psi_1'(y)\, F_{0,1}(0, 1)] + \phi_0(y)\, F_{1,0}(0, 0) \\ & + \phi_1(y)\left[-F_{0,1}(0, 0) + \left(\frac{\partial F_{0,1}(x, 0)}{\partial x}\right)_{x=0}\right] \\ & + \psi_0(y)\, [F_{1,0}(0, 1) - F_{0,1}(0, 1)] \\ & + \psi_1(y)\left[-F_{0,1}(0, 1) + \left(\frac{\partial F_{0,1}(x, 1-x)}{\partial x}\right)_{x=0}\right]. \end{aligned}$$

The remainder function

$$R = F - (P_1 \oplus P_2)F \tag{3.3}$$

is zero and has zero first derivatives on ∂T, except on $y = 0$, where

$$R_{0,1}(x, 0) = -\phi_1(x)\left[\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\right)(0, 0) - \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\right)(0, 0)\right]. \tag{3.4}$$

For this case it can be shown that

$$\begin{aligned} (P_2 \oplus P_3)R &= (P_3 \oplus P_2)R \\ &= -\frac{x^2 y(x+y-1)^2}{x+y}\left[\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\right)(0, 0) - \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\right)(0, 0)\right]. \end{aligned} \tag{3.5}$$

The function (3.5) clearly has the desired properties that it is zero and its first derivatives are zero on ∂T, except on $y = 0$, where its y partial derivative gives (3.4). Thus

$$(P_1 \oplus P_2)F + (P_3 \oplus P_2)R \tag{3.6}$$

gives the compatibly corrected interpolant to $F \in C^1(\partial T)$. The precision set of this interpolant contains the set of all polynomials of degree 5 and less.

The average of (3.6) and its dual for $(P_2 \oplus P_1)F$ gives a more symmetric form on the standard triangle.

4. Interpolation to Discrete Boundary Data

Smooth interpolation functions can be used to construct interpolation schemes which involve only point functionals on ∂T. (See, for example, Barnhill and Gregory [3], and Birkhoff and Mansfield [5].) This is achieved by defining the function and normal derivatives along a side of the triangle T as polynomials which interpolate the functionals on that side. This ensures continuity of the function and derivatives across a side common to two adjacent triangles. The requirement that $F \in C^{N-1}(\partial T)$, and that the interpolant on each triangle is defined by data on that triangle, implies that the function F and its partial derivatives of order $N - 1$ and less must be specified at each vertex. These $3N(N+1)/2$ values are sufficient for the modified smooth interpolation schemes of Theorem 2.4, but the compatibility conditions of the unmodified scheme would impose some additional higher-order partial derivatives at each vertex. The nth normal derivative along a side can then be defined as the polynomial which interpolates the $2(N - n)$ values given at the ends of the side, i.e., the nth normal derivative, and the $N - n - 1$ directional derivatives along the side of this nth normal derivative. The precision of the final interpolation function is limited by the lowest precision achieved by the boundary data interpolants and higher precision can be gained by including more point functionals along the sides.

As an example, we consider the smooth C^1 interpolant of Section 3. Defining the boundary functions of this interpolant as the cubic polynomials which interpolate F, and its first directional derivative along the side, at each vertex, and the normal derivatives as the quadratic polynomials which interpolate the normal derivative at the midpoints of the sides and at each vertex, results in a 12-parameter interpolation function with cubic precision. Alternatively, the normal derivative can be defined as the linear polynomials which interpolate the normal derivative at each vertex, or equivalently a linear variation on the normal derivative of the previous scheme can be imposed. This gives a nine-parameter interpolant with quadratic precision.

It is important to note that the interpolation schemes for discrete boundary data are not affine invariant and so cannot be treated only on the standard triangle. This follows since no direction is invariant under each of the affine transformations which take two adjacent triangles onto the standard one, except the direction along the common side. In particular, the normal derivative on the common side will in general be transferred into two different and nonnormal directions by each of these affine transformations.

References

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